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Monotone Iterative Method for Weakly Coupled System of Finite Difference Degenerate Parabolic Equations with Mixed Monotonicity Condition

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Abstract. The purpose of this paper is to present an existence-comparison and uniqueness theorem as well as an iterative method for finite difference system, which correspond to a class of semilinear weakly coupled system of time degenerate parabolic initial boundary value problems. The basic idea of the iterative method for the computation of numerical solutions is the monotone approach which involves the notion of upper and lower solutions and construction of monotone sequences from a suitable linear discrete system. Using upper and lower solutions as two distinct initial iterations, two monotone sequences are constructed from a suitable linear system under mixed quasimonotonicity condition. It is shown that these sequences converge monotonically to a solution of the discrete system.

Keywords: Monotone iterative method; Finite difference equations; Upper and lower solutions; Dirichlet initial boundary value problem; Existence-comparison and uniqueness; Mixed monotonicity.

1. Introduction

The method of upper lower solutions is one of the well known method employed successfully in the study of existence-comparison and uniqueness of solutions of initial boundary value problem (IBVP) for nonlinear partial differential equations. Certainly, this method is constructive and elegant. In the process of iteration both upper as well as lower bounds of the solution are in our hand. In 1985, Pao [12] introduced monotone method for finite difference equations of nonlinear parabolic and elliptic boundary value problems. In 2009, [9] study a behavior of the solution of the Cauchy problem for a parabolic equation under the condition of degeneration of the diffusion matrix. A series of papers appeared in the literature for reaction diffusion problems under different conditions (see [5, 6, 14, 15, 18]) and references therein. Dhaigude et al. [3] developed monotone scheme for the discrete Dirichlet IBVP which correspond to semilinear time degenerate parabolic Dirichlet IBVP and for weakly coupled system of finite difference equations which corresponds to weakly coupled system of semilinear time degenerate parabolic Dirichlet IBVP respectively. A series of papers appeared in the literature for monotone iterative method (see [1, 10, 16]). The monotone method for nonlinear partial differential equations discussed in [11, 17, 19]. In this paper, the backward approximation for the spatial derivative terms are used and the monotone scheme is developed, using the notion of upper-lower solutions for weakly coupled system of finite difference equations which corresponds to weakly coupled system of semilinear time degenerate parabolic Dirichlet IBVP when the reaction functions $f^{(1)}_{i,n}(u_{i,n}, v_{i,n})$ and $f^{(2)}_{i,n}(u_{i,n}, v_{i,n})$ are assumed to be mixed quasimonotone. Using upper and lower solutions as distinct initial iterations, two monotone convergent sequences from linear system are constructed. It is shown that these two sequences converge monotonically from above and below to maximal and minimal solutions respectively which lead to the existence-comparison and uniqueness results for the solution of the discrete Dirichlet IBVP. Positivity lemma is the main ingredient used in the proof of these results.

We organize the paper as follows: In section 2, Dirichlet IBVP for finite difference system is formulated from the corresponding continuous semilinear problem under consideration. The notion of upper lower solution is introduced. Section 3 is devoted for the construction of monotone scheme for the discrete Dirichlet IBVP. Using upper and lower solutions as distinct initial iterations, two monotone convergent sequences are constructed, which converge monotonically from above and below to maximal and minimal solutions respectively. Existence-comparison and uniqueness results for the solution of discrete Dirichlet IBVP are proved in the last section.

2. Finite Difference Equations

In this section, we obtain the discrete version of the Dirichlet initial boundary value problem for weakly coupled system of semilinear time degenerate parabolic equations. We consider the time degenerate Dirichlet initial boundary value Problem (IBVP) for weakly coupled system of semilinear time degenerate parabolic equations

$$\begin{aligned} d^{(1)}(x,t)u_t - D^{(1)}(x,t)\nabla^2 u = f^{(1)}(x,t,u,v); \\ d^{(2)}(x,t)v_t - D^{(2)}(x,t)\nabla^2 v = f^{(2)}(x,t,u,v); \end{aligned} \quad in\ D_T \tag{1}$$

Boundary conditions

$$\begin{aligned} u(x,t) &= h^{(1)}(x,t); \\ v(x,t) &= h^{(2)}(x,t); \end{aligned} \quad on\ S_T \tag{2}$$

Initial conditions

$$\begin{aligned} u(x,0) &= \psi^{(1)}(x); \\ v(x,0) &= \psi^{(2)}(x); \end{aligned} \quad in\ \Omega \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^p, (p = 1, 2, \dots)$ with boundary $\partial\Omega$. Here we denote parabolic domain and parabolic boundary by

$$D_T := \Omega \times (0, T]\ , S_T := \partial\Omega \times (0, T], T > 0 \text{ respectively.}$$

Suppose that

(i) The functions $d^{(1)}(x,t), d^{(2)}(x,t)$ are nonnegative in D_T. However we will not assume that $d^{(1)}(x,t)$ and $d^{(2)}(x,t)$ are bounded away from zero. Since both the coefficients $d^{(1)}(x,t) = 0, d^{(2)}(x,t) = 0$ for some $(x,t) \in D_T$ and hence the system is time degenerate.
(ii) $D^{(1)}(x,t) > 0, D^{(2)}(x,t) > 0$ in D_T.
(iii) the functions $f^{(1)}(x,t,u,v), f^{(2)}(x,t,u,v)$ are in general nonlinear in u, v and depend explicitly on (x,t). They are assumed to be mixed quasimonotone.

Now, we write the discrete version of the above continuous Dirichlet IBVP (1)-(3) by converting it into finite difference equations as in [1, 6, 7]. Let $i = (i_1, i_2,\ ...,\ i_p)$ be a multiple index with $i_\nu = 0, 1, 2, ..., M_\nu + 1$ and let $x_i = (x_{i_1}, x_{i_2}, ..., x_{i_p})$ be an arbitrary mesh point in Ω_p where M_ν is the total number of interior mesh points in the x_{i_ν} co-ordinate direction. Denoted by Ω_p, $\bar{\Omega}_p$, $\partial\Omega_p$, Λ_p and S_p the sets of mesh points in Ω, $\bar{\Omega}$, $\partial\Omega$, $\Omega \times (0,T)$ and $\partial\Omega \times (0,T)$ respectively and $\bar{\Lambda}_p$ denote the set of all mesh points in $\bar{\Omega} \times [0,T]$ where $\bar{\Omega}$ is the closure of Ω. Let (i,n) be used to represent the mesh point (x_i,t_n). Set

$$\begin{aligned}
u_{i,n} &\equiv u(x_i,t_n), v_{i,n} \equiv v(x_i,t_n), \\
d^{(1)}_{i,n} &\equiv d^{(1)}(x_i,t_n), d^{(2)}_{i,n} \equiv d^{(2)}(x_i,t_n) \\
D^{(1)}_{i,n} &\equiv D^{(1)}(x_i,t_n), D^{(2)}_{i,n} \equiv D^{(2)}(x_i,t_n), \\
h^{(1)}_{i,n} &\equiv h^{(1)}(x_i,t_n), h^{(2)}_{i,n} \equiv h^{(2)}(x_i,t_n) \\
u^{(1)}_{i,0} &\equiv u^{(1)}(x_i,0), u^{(2)}_{i,0} \equiv u^{(2)}(x_i,0) \\
v^{(1)}_{i,0} &\equiv v^{(1)}(x_i,0), v^{(2)}_{i,0} \equiv v^{(2)}(x_i,0) \\
\psi^{(1)}_{i} &\equiv \psi^{(1)}(x_i), \psi^{(2)}_{i} \equiv \psi^{(2)}(x_i) \\
f^{(1)}_{i,n}(u_{i,n},v_{i,n}) &\equiv f^{(1)}(x_i,t_n,u_{i,n},v_{i,n}), \\
f^{(2)}_{i,n}(u_{i,n},v_{i,n}) &\equiv f^{(2)}(x_i,t_n,u_{i,n},v_{i,n}).
\end{aligned}$$

Let $k_n = t_n - t_{n-1}$ be the n^{th} time increment for $n = 1, 2, ..., N$ and h_ν be the spatial increment in the x_{i_ν} co-ordinate direction. Let e_ν= (0,...,1...,0) be the unit vector in $\mathbb{R}^p$ where the constant 1 appears in the ν^{th} component and zero elsewhere. The standard second order difference approximation is [2, 8, 7]

$$\triangle^{(\nu)} u(x_i, t_n) = h_\nu^{(-2)}[u(x_i + h_\nu e_\nu, t_n) - 2u(x_i, t_n) + u(x_i - h_\nu e_\nu, t_n)] \tag{4}$$

and usual backward difference approximation for u_t by $k_n^{-1}(u_{i,n} - u_{i,n-1})$. Then the continuous Dirichlet IBVP (1)-(3) becomes

$$\begin{aligned} d_{i,n}^{(1)} k_n^{-1}(u_{i,n} - u_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} u_{i,n} = f_{i,n}^{(1)}(u_{i,n}, v_{i,n}); \\ d_{i,n}^{(2)} k_n^{-1}(v_{i,n} - v_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} v_{i,n} = f_{i,n}^{(2)}(u_{i,n}, v_{i,n}); \end{aligned} \quad (i,n) \in \Lambda_p \tag{5}$$

boundary conditions

$$\begin{aligned} u_{i,n} = h_{i,n}^{(1)}; \\ v_{i,n} = h_{i,n}^{(2)}; \end{aligned} \quad (i,n) \ \in \ S_p \tag{6}$$

initial conditions

$$\begin{aligned} u_{i,0} = \Psi_i^{(1)}; \\ v_{i,0} = \Psi_i^{(2)}; \end{aligned} \quad i \ \in \ \Omega_p \tag{7}$$

In this way we have obtained the discrete Dirichlet IBVP (5)-(7) for time degenerate parabolic partial differential equations. The reaction functions $f_{i,n}^{(1)}(u_{i,n}, v_{i,n})$ and $f_{i,n}^{(2)}(u_{i,n}, v_{i,n})$ are assumed to be mixed quasimonotone. In this case, we consider that $f_{i,n}^{(1)}(u_{i,n}, v_{i,n})$ is quasimonotone nonincreasing and $f_{i,n}^{(2)}(u_{i,n}, v_{i,n})$ is quasimonotone nondecreasing. We define it as follows

Definition 2.1. *The C^1-functions $f_{i,n}^{(1)}$and$f_{i,n}^{(2)}$ are said to be mixed quasimonotone if*

$$f_{i,n}^{(1)}(u_{i,n}, v_{i,n}) \geq f_{i,n}^{(1)}(u_{i,n}, v'_{i,n}) \textit{ for } v'_{i,n} \geq v_{i,n}$$

and

$$f_{i,n}^{(2)}(u_{i,n}, v_{i,n}) \leq f_{i,n}^{(2)}(u'_{i,n}, v_{i,n}) \textit{ for } u'_{i,n} \geq u_{i,n}$$

respectively (or vice versa).

3. Upper Lower Solutions

We develop monotone scheme for the system of finite difference time degenerate Dirichlet IBVP (5)-(7). The discrete version of the positivity lemma in [4] for the continuous problem play an important role in the construction of monotone and convergent sequences. Now we develop monotone scheme for time degenerate discrete Dirichlet IBVP (5)-(7). We define upper and lower solutions of the time degenerate discrete Dirichlet IBVP (5)-(7).

Definition 3.1. *The functions* $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ *and* $(\hat{u}_{i,n}, \hat{v}_{i,n})$ *with* $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \geq (\hat{u}_{i,n}, \hat{v}_{i,n})$ *are called ordered upper and lower solutions of time degenerate discrete Dirichlet IBVP* (5)-(7) *if they satisfy differential inequalities*

$$\left.\begin{aligned} d^{(1)}_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\upsilon=1}^{p} D^{(1)}_{i,n}\triangle^{(\upsilon)}\tilde{u}_{i,n} &\geq f^{(1)}_{i,n}(\tilde{u}_{i,n}, \hat{v}_{i,n}); \\ d^{(1)}_{i,n}k_n^{-1}(\hat{u}_{i,n} - \hat{u}_{i,n-1}) - \sum_{\upsilon=1}^{p} D^{(1)}_{i,n}\triangle^{(\upsilon)}\hat{u}_{i,n} &\leq f^{(1)}_{i,n}(\hat{u}_{i,n}, \tilde{v}_{i,n}); \\ d^{(2)}_{i,n}k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - \sum_{\upsilon=1}^{p} D^{(2)}_{i,n}\triangle^{(\upsilon)}\tilde{v}_{i,n} &\geq f^{(2)}_{i,n}(\tilde{u}_{i,n}, \tilde{v}_{i,n}); \\ d^{(2)}_{i,n}k_n^{-1}(\hat{v}_{i,n} - \hat{v}_{i,n-1}) - \sum_{\upsilon=1}^{p} D^{(2)}_{i,n}\triangle^{(\upsilon)}\hat{v}_{i,n} &\leq f^{(2)}_{i,n}(\hat{u}_{i,n}, \hat{v}_{i,n}); \end{aligned}\right\} \quad (i,n) \in \Lambda_p$$

Boundary conditions

$$\left.\begin{aligned} \tilde{u}_{i,n} &\geq h^{(1)}_{i,n}; \\ \hat{u}_{i,n} &\leq h^{(1)}_{i,n} \\ \tilde{v}_{i,n} &\geq h^{(2)}_{i,n}; \\ \hat{v}_{i,n} &\leq h^{(2)}_{i,n} \end{aligned}\right\} \quad (i,n) \in S_p$$

Initial conditions

$$\left.\begin{aligned} \tilde{u}_{i,0} &\geq \Psi^{(1)}_{i}; \\ \hat{u}_{i,0} &\leq \Psi^{(1)}_{i} \\ \tilde{v}_{i,0} &\geq \Psi^{(2)}_{i}; \\ \hat{v}_{i,0} &\leq \Psi^{(2)}_{i}. \end{aligned}\right\} \quad i \in \Omega_p$$

Definition 3.2. *For any ordered upper and lower solutions* $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ *and* $(\hat{u}_{i,n}$ $,\hat{v}_{i,n})$ *the sector is denoted by* $S_{i,n}$ *and is defined as*

$$S_{i,n} = \{(u_{i,n}, v_{i,n}) \in \bar{\Lambda}_p : (\hat{u}_{i,n}, \hat{v}_{i,n}) \leq (u_{i,n}, v_{i,n}) \leq (\tilde{u}_{i,n}, \tilde{v}_{i,n})\}. \quad (8)$$

Suppose there exist nonnegative constants $c_{i,n}^{(1)}$ and $c_{i,n}^{(2)}$ such that the function $(f_{i,n}^{(1)}, f_{i,n}^{(2)})$ satisfies the following one sided Lipschitz condition

$$\begin{aligned} f_{i,n}^{(1)}(u_{i,n}, v_{i,n}) - f_{i,n}^{(1)}(u'_{i,n}, v_{i,n}) &\geq -c_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \text{ when } u_{i,n} \geq u'_{i,n} \\ f_{i,n}^{(2)}(u_{i,n}, v_{i,n}) - f_{i,n}^{(2)}(u_{i,n}, v'_{i,n}) &\geq -c_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \text{ when } v_{i,n} \geq v'_{i,n}. \end{aligned} \tag{9}$$

Further more, for any $(u_{i,n}, v_{i,n})$ and $(u'_{i,n}, v'_{i,n})$ in the sector $S_{i,n}$,we have

$$\begin{aligned} f_{i,n}^{(1)}(u_{i,n}, v_{i,n}) - f_{i,n}^{(1)}(u'_{i,n}, v_{i,n}) &\leq c_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \text{ when } u_{i,n} \geq u'_{i,n} \\ f_{i,n}^{(2)}(u_{i,n}, v_{i,n}) - f_{i,n}^{(2)}(u_{i,n}, v'_{i,n}) &\leq c_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \text{ when } v_{i,n} \geq v'_{i,n}. \end{aligned} \tag{10}$$

Adding $c_{i,n}^{(1)}u_{i,n}$ and $c_{i,n}^{(2)}v_{i,n}$ on both sides of equation in (5) respectively, we get

$$\begin{aligned} d_{i,n}^{(1)}k_n^{-1}(u_{i,n} - u_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)}\triangle^{(\nu)}u_{i,n} + c_{i,n}^{(1)}u_{i,n} &= c_{i,n}^{(1)}u_{i,n} + f_{i,n}^{(1)}(u_{i,n}, v_{i,n}) \\ d_{i,n}^{(2)}k_n^{-1}(v_{i,n} - v_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)}\triangle^{(\nu)}v_{i,n} + c_{i,n}^{(2)}v_{i,n} &= c_{i,n}^{(2)}v_{i,n} + f_{i,n}^{(2)}(u_{i,n}, v_{i,n}). \end{aligned}$$

Suppose

$$\begin{aligned} L[u_{i,n}] &\equiv d_{i,n}^{(1)}k_n^{-1}(u_{i,n} - u_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)}\triangle^{(\nu)}u_{i,n} + c_{i,n}^{(1)}u_{i,n} \\ L[v_{i,n}] &\equiv d_{i,n}^{(2)}k_n^{-1}(v_{i,n} - v_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)}\triangle^{(\nu)}v_{i,n} + c_{i,n}^{(2)}v_{i,n} \end{aligned}$$

and

$$\begin{aligned} L[u_{i,n}^{m}] &\equiv d_{i,n}^{(1)}k_n^{-1}(u_{i,n}^{(m)} - u_{i,n-1}^{(m)}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)}\triangle^{(\nu)}u_{i,n}^{(m)} + c_{i,n}^{(1)}u_{i,n}^{(m)} \\ L[v_{i,n}^{m}] &\equiv d_{i,n}^{(2)}k_n^{-1}(v_{i,n}^{(m)} - v_{i,n-1}^{(m)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)}\triangle^{(\nu)}{v^{(m)}}_{i,n} + c_{i,n}^{(2)}v_{i,n}^{(m)}. \end{aligned}$$

Lemma 3.3. (Positivity Lemma) [4] *Suppose that $w_{i,n}$ satisfies the following inequalities*

$$\begin{aligned} &d_{i,n}k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D_{i,n}\Delta^{(\nu)}w_{i,n} + c_{i,n}w_{i,n} \geq 0; (i,n) \in \Lambda_p, \\ &Bw_{i,n} \geq 0; (i,n) \in S_p, \\ &w_{i,0} \geq 0; i \in \Omega_p, \end{aligned}$$

where $Bw_{i,n} = \alpha(x_i,t_n)|x_i - \hat{x}_i|^{-1}[w(x_i,t_n) - w(\hat{x}_i,t_n)] + \beta(x_i,t_n)\frac{\partial w}{\partial \nu}(x_i,t_n)$, $c_{i,n} \geq 0, d_{i,n} \geq 0, \hat{x}_i$ *is a suitable point in* Ω_p *and* $|x_i - \hat{x}_i|$ *is the distance between* x_i *and* $\hat{x}_i$ *then* $w_{i,n} \geq 0$ *in* $\bar{\Lambda}_p$.

4. Monotone Iterative Scheme

We choose suitable initial iterations $(u^{(0)}_{i,n}, v^{(0)}_{i,n})$ as either $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ or $(\hat{u}_{i,n}, \hat{v}_{i,n})$ and construct a sequence of iterations $\{u^{(m)}_{i,n}, v^{(m)}_{i,n}\}$ from the following iteration processes

$$\begin{aligned} L[u^{(m)}_{i,n}] &= c^{(1)}_{i,n} u^{(m-1)}_{i,n} + f^{(1)}_{i,n}(u^{(m-1)}_{i,n}, v^{(m-1)}_{i,n}), \\ L[v^{(m)}_{i,n}] &= c^{(2)}_{i,n} v^{(m-1)}_{i,n} + f^{(2)}_{i,n}(u^{(m)}_{i,n}, v^{(m-1)}_{i,n}), \end{aligned} \qquad (i,n) \in \Lambda_p$$

$$\begin{aligned} u^{(m)}_{i,n} &= h^{(1)}_{i,n} \\ v^{(m)}_{i,n} &= h^{(2)}_{i,n}, \end{aligned} \qquad (i,n) \in S_p$$

$$\begin{aligned} u^{(m)}_{i,0} &= \psi^{(1)}_i \\ v^{(m)}_{i,0} &= \psi^{(2)}_i. \end{aligned} \qquad i \in \Omega_p$$

It is a system of linear algebraic equations. Here $m = 1$ and with suitable choice of initial iterations $(\bar{u}^{(0)}_{i,n}, \underline{v}^{(0)}_{i,n}) = (\tilde{u}_{i,n}, \hat{v}_{i,n})$. we get $(\bar{u}^{(1)}_{i,n})$ from first equation. Put this value in second equation, we get first iteration $(\bar{u}^{(1)}_{i,n}, \underline{v}^{(1)}_{i,n})$. Repeat the process for $m = 2, 3, ...$ We construct a sequence $\{\bar{u}^{(m)}_{i,n}, \underline{v}^{(m)}_{i,n}\}$. Further, suppose $m = 1$ and with suitable choice of initial iterations $(\underline{u}^{(0)}_{i,n}, \bar{v}^{(0)}_{i,n}) = (\hat{u}_{i,n}, \tilde{v}_{i,n})$. we get $(\underline{u}^{(1)}_{i,n})$ from first equation. Put this value in second equation, we get first iteration $(\underline{u}^{(1)}_{i,n}, \bar{v}^{(1)}_{i,n})$. Repeat the process for m=2,3,..... We construct a sequence $\{\underline{u}^{(m)}_{i,n}, \bar{v}^{(m)}_{i,n}\}$.

Furthermore, we note an interesting point about the above iterative scheme is that the component $(\bar{u}^{(1)}_{i,n})$ from first equation and is used immediately in second equation and $(\bar{v}^{(1)}_{i,n})$ is obtained. We note the above kind of iteration is similar to the Gauss-Seidal iterative method for algebraic systems. It has the advantage of obtaining faster convergent sequences.

In the following, we obtain the monotone property of the sequences when initial iteration is either an upper solution or a lower solution.

Lemma 4.1. [Monotone Property] *Suppose that*

(i) $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ *and* $(\hat{u}_{i,n}, \hat{v}_{i,n})$ *are ordered upper and lower solutions of discrete Dirichlet IBVP* (5)-(7).

(ii) *a function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *is mixed quasimonotone in* $S_{i,n}$.

(iii) *the function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *satisfies the following one sided Lipschitz condition*

$$\begin{aligned} f^{(1)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(1)}_{i,n}(u'_{i,n}, v_{i,n}) &\geq -c^{(1)}_{i,n}(u_{i,n} - u'_{i,n}) \text{ if } u_{i,n} \geq u'_{i,n} \\ f^{(2)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(2)}_{i,n}(u_{i,n}, v'_{i,n}) &\geq -c^{(2)}_{i,n}(v_{i,n} - v'_{i,n}) \text{ if } v_{i,n} \geq v'_{i,n}. \end{aligned} \tag{11}$$

Then the sequences $\{\bar{u}^{(m)}_{i,n}, \bar{v}^{(m)}_{i,n}\}$ *and* $\{\underline{u}^{(m)}_{i,n}, \underline{v}^{(m)}_{i,n}\}$ *possess the monotone property,*

$$\hat{u}_{i,n} \leq \underline{u}^{(1)}_{i,n} \leq \cdots \leq \underline{u}^{(m+1)}_{i,n} \leq \bar{u}^{(m+1)}_{i,n} \leq \cdots \leq \bar{u}^{(1)}_{i,n} \leq \tilde{u}_{i,n} \text{ in } \bar{\Lambda}_p \tag{12}$$

$$\hat{v}_{i,n} \leq \underline{v}^{(1)}_{i,n} \leq \cdots \leq \underline{v}^{(m+1)}_{i,n} \leq \bar{v}^{(m+1)}_{i,n} \leq \cdots \leq \bar{v}^{(1)}_{i,n} \leq \tilde{v}_{i,n} \text{ in } \bar{\Lambda}_p. \tag{13}$$

Proof. Define $w_{i,n} = \bar{u}^{(0)}_{i,n} - \bar{u}^{(1)}_{i,n} = \tilde{u}_{i,n} - \bar{u}^{(1)}_{i,n}$, where $\bar{u}^{(0)}_{i,n} = \tilde{u}_{i,n}$. We have

$$\begin{aligned} L[w_{i,n}] &= L[\tilde{u}_{i,n}] - L[\bar{u}^{(1)}_{i,n}], \\ L[w_{i,n}] &= d^{(1)}_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}\tilde{u}_{i,n} + c^{(1)}_{i,n}\tilde{u}_{i,n} \\ &\quad -[d^{(1)}_{i,n}k_n^{-1}(\bar{u}^{(1)}_{i,n} - \bar{u}^{(1)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}\bar{u}^{(1)}_{i,n} + c^{(1)}_{i,n}\bar{u}^{(1)}_{i,n}] \\ &= d^{(1)}_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}\tilde{u}_{i,n} + c^{(1)}_{i,n}\tilde{u}_{i,n} \\ &\quad -[c^{(1)}_{i,n}\tilde{u}_{i,n} + f^{(1)}_{i,n}(\tilde{u}_{i,n}, \hat{v}_{i,n})] \qquad \text{(By iteration Process)} \\ &= d^{(1)}_{i,n}k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}\tilde{u}_{i,n} - f^{(1)}_{i,n}(\tilde{u}_{i,n}, \hat{v}_{i,n}) \\ &\geq 0 \qquad \text{(Since } \tilde{u}_{i,n} \text{ is an upper solution)} \end{aligned}$$

$$\begin{aligned} &d^{(1)}_{i,n}k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}w_{i,n} + c^{(1)}_{i,n}w_{i,n} \geq 0; (i,n) \in \Lambda_p \\ &w_{i,n} = \tilde{u}_{i,n} - h^{(1)}_{i,n} \geq 0; (i,n) \in S_p \\ &w_{i,0} = \tilde{u}_{i,0} - \Psi^{(1)}_{i} \geq 0; i \in \Omega_p \end{aligned}$$

Applying positivity Lemma 3.3, we get $w_{i,n} \geq 0$ implies that $\bar{u}^{(1)}_{i,n} \leq \tilde{u}_{i,n}$. Similarly, we can show that $\hat{u}_{i,n} \leq \underline{u}^{(1)}_{i,n}$. Define $z_{i,n} = \underline{v}^{(1)}_{i,n} - \underline{v}^{(0)}_{i,n} = \underline{v}^{(1)}_{i,n} - \hat{v}_{i,n}$, where $\underline{v}^{(0)}_{i,n} = \hat{v}_{i,n}$. We have

$$\begin{aligned} L[z_{i,n}] &= L[\underline{v}^{(1)}_{i,n}] - L[\hat{v}_{i,n}], \\ L[z_{i,n}] &= d^{(2)}_{i,n}k_n^{-1}(\underline{v}^{(1)}_{i,n} - \underline{v}^{(1)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(2)}_{i,n}\triangle^{(\nu)}\underline{v}^{(1)}_{i,n} + c^{(2)}_{i,n}\underline{v}^{(1)}_{i,n} \end{aligned}$$

$$
\begin{aligned}
&-[d_{i,n}^{(2)}k_n^{-1}(\hat{v}_{i,n}-\hat{v}_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}\hat{v}_{i,n}+c_{i,n}^{(2)}\hat{v}_{i,n}]\\
&= c_{i,n}^{(2)}\hat{v}_{i,n}+f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)},\hat{v}_{i,n})-[d_{i,n}^{(2)}k_n^{-1}(\hat{v}_{i,n}-\hat{v}_{i,n-1})\\
&\quad -\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}\hat{v}_{i,n}+c_{i,n}^{(2)}\hat{v}_{i,n}] \quad \text{(By iteration Process)}\\
&= f_{i,n}^{(2)}((\underline{u}_{i,n}^{(1)},\hat{v}_{i,n})-[d_{i,n}^{(2)}k_n^{-1}(\hat{v}_{i,n}-\hat{v}_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}\hat{v}_{i,n}]\\
&\qquad \text{(We know that } f_{i,n}^{(2)} \text{ is quasimonotone nondecreasing function)}\\
&\geq f_{i,n}^{(2)}(\hat{u}_{i,n},\hat{v}_{i,n})-[d_{i,n}^{(2)}k_n^{-1}(\hat{v}_{i,n}-\hat{v}_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}\hat{v}_{i,n}]\\
&\geq 0 \quad \text{(Since } \hat{v}_{i,n} \text{ is lower solution)}
\end{aligned}
$$

$$
\begin{aligned}
&d_{i,n}^{(2)}k_n^{-1}(z_{i,n}-z_{i,n-1})-\sum_{\nu=1}^{p}D_{i,n}^{(2)}\triangle^{(\nu)}z_{i,n}+c_{i,n}^{(2)}z_{i,n}\geq 0;(i,n)\in\Lambda_p\\
&z_{i,n}=\underline{v}_{i,n}^{(1)}-\hat{v}_{i,n}\geq h_{i,n}^{(2)}-\hat{v}_{i,n}\geq 0;(i,n)\in S_p\\
&z_{i,0}=\underline{v}_{i,0}^{(1)}-\hat{v}_{i,0}\geq \Psi_i^{(2)}-\hat{v}_{i,0}\geq 0;i\in\Omega_p
\end{aligned}
$$

Applying positivity Lemma 3.3, we get $z_{i,n}\geq 0$ implies that $\underline{v}_{i,n}^{(1)}\geq\hat{v}_{i,n}$.

On similar lines, using the property of $(\hat{u}_{i,n},\tilde{v}_{i,n})$ gives

$$\underline{u}_{i,n}^{(1)}\geq\hat{u}_{i,n} \quad \text{and} \quad \bar{v}_{i,n}^{(1)}\leq\tilde{v}_{i,n}.$$

Next we define $w_{i,n}^{(1)}=\bar{u}_{i,n}^{(1)}-\underline{u}_{i,n}^{(1)}$. We have

$$
\begin{aligned}
L[w_{i,n}^{(1)}] &= L[\bar{u}_{i,n}^{(1)}]-L[\underline{u}_{i,n}^{(1)}],\\
L[w_{i,n}^{(1)}] &= d_{i,n}^{(1)}k_n^{-1}(\bar{u}_{i,n}^{(1)}-\bar{u}_{i,n-1}^{(1)})-\sum_{\nu=1}^{p}D_{i,n}^{(1)}\triangle^{(\nu)}\bar{u}_{i,n}^{(1)}+c_{i,n}^{(1)}\bar{u}_{i,n}^{(1)}\\
&\quad -[d_{i,n}^{(1)}k_n^{-1}(\underline{u}_{i,n}^{(1)}-\underline{u}_{i,n-1}^{(1)})-\sum_{\nu=1}^{p}D_{i,n}^{(1)}\triangle^{(\nu)}\underline{u}_{i,n}^{(1)}+c_{i,n}^{(1)}\underline{u}_{i,n}^{(1)}]\\
&= c_{i,n}^{(1)}\bar{u}_{i,n}^{(0)}+f_{i,n}^{(1)}(\bar{u}_{i,n}^{(0)},\underline{v}_{i,n}^{(0)})-c_{i,n}^{(1)}\underline{u}_{i,n}^{(0)}-f_{i,n}^{(1)}(\underline{u}_{i,n}^{(0)},\bar{v}_{i,n}^{(0)})\\
&\qquad \text{(By iteration Process)}\\
&= [c_{i,n}^{(1)}(\bar{u}_{i,n}^{(0)}-\underline{u}_{i,n}^{(0)})+f_{i,n}^{(1)}(\bar{u}_{i,n}^{(0)},\underline{v}_{i,n}^{(0)})-f_{i,n}^{(1)}(\underline{u}_{i,n}^{(0)},\underline{v}_{i,n}^{(0)})]\\
&\quad +[f_{i,n}^{(1)}(\underline{u}_{i,n}^{(0)},\underline{v}_{i,n}^{(0)})-f_{i,n}^{(1)}(\underline{u}_{i,n}^{(0)},\bar{v}_{i,n}^{(0)})]\\
&\geq 0 \quad (f_{i,n}^{(1)} \text{ is Lipchitzian and quasimonotone nonincreasing}),
\end{aligned}
$$

$$d_{i,n}^{(1)}k_n^{-1}(w_{i,n}^{(1)}-w_{i,n-1}^{(1)})-\sum_{\nu=1}^{p}D_{i,n}^{(1)}\triangle^{(\nu)}w_{i,n}^{(1)}+c_{i,n}^{(1)}w_{i,n}^{(1)}\geq 0;(i,n)\in\Lambda_p$$

$$w_{i,n}^{(1)} = \bar{u}_{i,n}^{(1)} - \underline{u}_{i,n}^{(1)} \geq h_{i,n}^{(1)} - \underline{u}_{i,n}^{(1)} \geq 0; (i,n) \in S_p$$
$$w_{i,0}^{(1)} = \bar{u}_{i,0}^{(1)} - \underline{u}_{i,0}^{(1)} \geq \Psi_i^{(1)} - \underline{u}_{i,0}^{(1)} \geq 0; i \in \Omega_p.$$

Applying positivity Lemma 3.3, we get $w_{i,n}^{(1)} \geq 0$ implies that $\underline{u}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(1)}$.

Define $z_{i,n}^{(1)} = \bar{v}_{i,n}^{(1)} - \underline{v}_{i,n}^{(1)}$. We have

$$\begin{aligned}
L[z_{i,n}^{(1)}] &= L[\bar{v}_{i,n}^{(1)}] - L[\underline{v}_{i,n}^{(1)}] \\
L[z_{i,n}^{(1)}] &= d_{i,n}^{(2)} k_n^{-1}(\bar{v}_{i,n}^{(1)} - \bar{v}_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} \bar{v}_{i,n}^{(1)} + c_{i,n}^{(2)} \bar{v}_{i,n}^{(1)} \\
&\quad -[d_{i,n}^{(2)} k_n^{-1}(\underline{v}_{i,n}^{(1)} - \underline{v}_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} \underline{v}_{i,n}^{(1)} + c_{i,n}^{(2)} \underline{v}_{i,n}^{(1)}] \\
&= c_{i,n}^{(2)} \bar{v}_{i,n}^{(0)} + f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \bar{v}_{i,n}^{(0)}) - c_{i,n}^{(2)} \underline{v}_{i,n}^{(0)} - f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \\
&\qquad \text{(By iteration Process)} \\
&= [c_{i,n}^{(2)}(\bar{v}_{i,n}^{(0)} - \underline{v}_{i,n}^{(0)}) + f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \bar{v}_{i,n}^{(0)}) - f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)})] \\
&\quad +[f_{i,n}^{(2)}(\bar{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) - f_{i,n}^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)})] \\
&\geq 0 \quad (f_{i,n}^{(2)} \text{ is Lipchitzian and quasimonotone nondecreasing})
\end{aligned}$$

$$d_{i,n}^{(2)} k_n^{-1}(z_{i,n}^{(1)} - z_{i,n-1}^{(1)}) - \sum_{\nu=1}^{p} D_{i,n}^{(2)} \triangle^{(\nu)} z_{i,n}^{(1)} + c_{i,n}^{(2)} z_{i,n}^{(1)} \geq 0; (i,n) \in \Lambda_p$$
$$z_{i,n}^{(1)} = \bar{v}_{i,n}^{(1)} - \underline{v}_{i,n}^{(1)} \geq h_{i,n}^{(2)} - \underline{v}_{i,n}^{(1)} \geq 0; (i,n) \in S_p$$
$$z_{i,0}^{(1)} = \bar{v}_{i,0}^{(1)} - \underline{v}_{i,0}^{(1)} \geq \Psi_i^{(2)} - \underline{v}_{i,0}^{(1)} \geq 0; i \in \Omega_p.$$

Applying positivity Lemma 3.3, we get $z_{i,n}^{(1)} \geq 0$ implies that $\underline{v}_{i,n}^{(1)} \leq \bar{v}_{i,n}^{(1)}$.

We conclude that

$$\hat{u}_{i,n} \leq \underline{\mathrm{u}}_{i,n}^{(1)} \leq \bar{u}_{i,n}^{(1)} \leq \tilde{u}_{i,n} \text{ in } \bar{\Lambda}_p$$
$$\hat{v}_{i,n} \leq \underline{v}_{i,n}^{(1)} \leq \bar{v}_{i,n}^{(1)} \leq \tilde{v}_{i,n} \text{ in } \bar{\Lambda}_p$$

Thus result is true for $m = 1$. Assume that it is true for $m = l$

$$\underline{\mathrm{u}}_{i,n}^{(l-1)} \leq \underline{\mathrm{u}}_{i,n}^{(l)} \leq \bar{u}_{i,n}^{(l)} \leq \bar{u}_{i,n}^{(l-1)} \text{ in } \bar{\Lambda}_p$$
$$\underline{v}_{i,n}^{(l-1)} \leq \underline{v}_{i,n}^{(l)} \leq \bar{v}_{i,n}^{(l)} \leq \bar{v}_{i,n}^{(l-1)} \text{ in } \bar{\Lambda}_p.$$

Now we prove that it is true for $m = l + 1$.

Define $w_{i,n}^{(l)} = \bar{u}_{i,n}^{(l)} - \bar{u}_{i,n}^{(l+1)}$. We have

$$\begin{aligned}
L[w_{i,n}^{(l)}] &= L[\bar{u}_{i,n}^{(l)}] - L[\bar{u}_{i,n}^{(l+1)}], \\
L[w_{i,n}^{(l)}] &= d_{i,n}^{(1)} k_n^{-1}(\bar{u}_{i,n}^{(l)} - \bar{u}_{i,n-1}^{(l)}) - \sum_{\nu=1}^{p} D_{i,n}^{(1)} \triangle^{(\nu)} \bar{u}_{i,n}^{(l)} + c_{i,n}^{(1)} \bar{u}_{i,n}^{(l)}
\end{aligned}$$

$$
\begin{aligned}
&\quad -[d^{(1)}_{i,n}k_n^{-1}(\bar{u}^{(l+1)}_{i,n} - \bar{u}^{(l+1)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}\bar{u}^{(l+1)}_{i,n} + c^{(1)}_{i,n}\bar{u}^{(l+1)}_{i,n}] \\
&= c^{(1)}_{i,n}\bar{u}^{(l-1)}_{i,n} + f^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n}, \underline{v}^{(l-1)}_{i,n}) - [c^{(1)}_{i,n}\bar{u}^{(l)}_{i,n} + f^{(1)}_{i,n}(\bar{u}^{(l)}_{i,n}, \underline{v}^{(l)}_{i,n})] \\
&\qquad \text{(By iteration Process)} \\
&= c^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n} - \bar{u}^{(l)}_{i,n}) + f^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n}, \underline{v}^{(l-1)}_{i,n}) - f^{(1)}_{i,n}(\bar{u}^{(l)}_{i,n}, \underline{v}^{(l)}_{i,n}) \\
&= [c^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n} - \bar{u}^{(l)}_{i,n}) + f^{(1)}_{i,n}(\bar{u}^{(l-1)}_{i,n}, \underline{v}^{(l-1)}_{i,n}) - f^{(1)}_{i,n}(\bar{u}^{(l)}_{i,n}, \underline{v}^{(l-1)}_{i,n})] \\
&\quad +[f^{(1)}_{i,n}(\bar{u}^{(l)}_{i,n}, \underline{v}^{(l-1)}_{i,n}) - f^{(1)}_{i,n}(\bar{u}^{(l)}_{i,n}, \underline{v}^{(l)}_{i,n})] \\
&\geq 0 \quad (f^{(1)}_{i,n} \text{ is Lipchitzian and quasimonotone nonincreasing})
\end{aligned}
$$

$$
\begin{aligned}
&d^{(1)}_{i,n}k_n^{-1}(w^{(l)}_{i,n} - w^{(l)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n}\triangle^{(\nu)}w^{(l)}_{i,n} + c^{(1)}_{i,n}w^{(l)}_{i,n} \geq 0; (i,n) \in \Lambda_p \\
&w^{(l)}_{i,n} = \bar{u}^{(l)}_{i,n} - \bar{u}^{(l+1)}_{i,n} \geq h^{(1)}_{i,n} - \underline{u}^{(l)}_{i,n} \geq 0; (i,n) \in S_p \\
&w^{(l)}_{i,0} = \bar{u}^{(l)}_{i,0} - \bar{u}^{(l+1)}_{i,0} \geq \Psi^{(1)}_{i} - \underline{u}^{(l)}_{i,0} \geq 0; i \in \Omega_p.
\end{aligned}
$$

Applying positivity Lemma 3.3, we get $w^{(l)}_{i,n} \geq 0$ implies that $\bar{u}^{(l+1)}_{i,n} \leq \bar{u}^{(l)}_{i,n}$. On similar lines we can prove that $\underline{u}^{(l)}_{i,n} \leq \underline{u}^{(l+1)}_{i,n}$ and $\underline{u}^{(l+1)}_{i,n} \leq \bar{u}^{(l+1)}_{i,n}$. Define $z^{(l)}_{i,n} = \bar{v}^{(l)}_{i,n} - \bar{v}^{(l+1)}_{i,n}$. We have

$$
\begin{aligned}
L[z^{(l)}_{i,n}] &= L[\bar{v}^{(l)}_{i,n}] - L[\bar{v}^{(l+1)}_{i,n}], \\
L[z^{(l)}_{i,n}] &= d^{(2)}_{i,n}k_n^{-1}(\bar{v}^{(l)}_{i,n} - \bar{v}^{(l)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(2)}_{i,n}\triangle^{(\nu)}\bar{v}^{(l)}_{i,n} + c^{(2)}_{i,n}\bar{v}^{(l)}_{i,n} \\
&\quad -[d^{(2)}_{i,n}k_n^{-1}(\bar{v}^{(l+1)}_{i,n} - \bar{v}^{(l+1)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(2)}_{i,n}\triangle^{(\nu)}\bar{v}^{(l+1)}_{i,n} + c^{(2)}_{i,n}\bar{v}^{(l+1)}_{i,n}] \\
&= c^{(2)}_{i,n}\bar{v}^{(l-1)}_{i,n} + f^{(2)}_{i,n}(\bar{u}^{(l)}_{i,n}, \bar{v}^{(l-1)}_{i,n}) - c^{(2)}_{i,n}\bar{v}^{(l)}_{i,n} - f^{(2)}_{i,n}(\bar{u}^{(l+1)}_{i,n}, \bar{v}^{(l)}_{i,n}) \\
&\qquad \text{(By iteration Process)} \\
&= [c^{(2)}_{i,n}(\bar{v}^{(l-1)}_{i,n} - \bar{v}^{(l)}_{i,n}) + f^{(2)}_{i,n}(\bar{u}^{(l)}_{i,n}, \bar{v}^{(l-1)}_{i,n}) - f^{(2)}_{i,n}(\bar{u}^{(l)}_{i,n}, \bar{v}^{(l)}_{i,n})] \\
&\quad +[f^{(2)}_{i,n}(\bar{u}^{(l)}_{i,n}, \bar{v}^{(l)}_{i,n}) - f^{(2)}_{i,n}(\bar{u}^{(l+1)}_{i,n}, \bar{v}^{(l)}_{i,n})] \\
&\geq 0 \quad (f^{(2)}_{i,n} \text{ is Lipchitzian and quasimonotone nondecreasing})
\end{aligned}
$$

$$
\begin{aligned}
&d^{(2)}_{i,n}k_n^{-1}(z^{(l)}_{i,n} - z^{(l)}_{i,n-1}) - \sum_{\nu=1}^{p} D^{(2)}_{i,n}\triangle^{(\nu)}z^{(l)}_{i,n} + c^{(2)}_{i,n}z^{(l)}_{i,n} \geq 0; (i,n) \in \Lambda_p \\
&z^{(l)}_{i,n} = \bar{v}^{(l)}_{i,n} - \bar{v}^{(l+1)}_{i,n} \geq \bar{v}^{(l)}_{i,n} - h^{(2)}_{i,n} \geq 0; (i,n) \in S_p \\
&z^{(l)}_{i,0} = \bar{v}^{(l)}_{i,0} - \bar{v}^{(l+1)}_{i,0} \geq \bar{v}^{(l)}_{i,0} - \Psi^{(2)}_{i} \geq 0; i \in \Omega_p.
\end{aligned}
$$

Applying positivity Lemma 3.3, we get $w^{(l)}_{i,n} \geq 0$ implies that $\bar{v}^{(l+1)}_{i,n} \leq \bar{v}^{(l)}_{i,n}$. Note that using similar arguments we can prove $\underline{v}^{(l)}_{i,n} \leq \underline{v}^{(l+1)}_{i,n}$ as well as $\underline{v}^{(l+1)}_{i,n} \leq \bar{v}^{(l+1)}_{i,n}$. Thus from principle of mathematical induction result is true for all m. ∎

5. Applications

In this section, we show that the monotone method can be successfully applied to prove existence - comparison and uniqueness results for the solution of the discrete time degenerate Dirichlet IBVP (5)-(7).

Theorem 5.1. (Existence-Comparison Theorem) *Suppose that*

(i) $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ *and* $(\hat{u}_{i,n}, \hat{v}_{i,n})$ *are ordered upper and lower solutions of discrete time degenerate Dirichlet IBVP* (5)-(7).

(ii) *a function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *is mixed quasimonotone in* $S_{i,n}$,

(iii) *the function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *satisfies the following one sided Lipschitz condition*

$$f^{(1)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(1)}_{i,n}(u'_{i,n}, v_{i,n}) \geq -c^{(1)}_{i,n}(u_{i,n} - u'_{i,n}) \text{ when } u_{i,n} \geq u'_{i,n}$$
$$f^{(2)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(2)}_{i,n}(u_{i,n}, v'_{i,n}) \geq -c^{(2)}_{i,n}(v_{i,n} - v'_{i,n}) \text{ when } v_{i,n} \geq v'_{i,n}$$

Then the sequence $\{\bar{u}^{(m)}_{i,n}, \bar{v}^{(m)}_{i,n}\}$ *converges monotonically from above to maximal solution* $\{\bar{u}_{i,n}, \bar{v}_{i,n}\}$ *and the sequence* $\{\underline{u}^{(m)}_{i,n}, \underline{v}^{(m)}_{i,n}\}$ *converges monotonically from below to minimal solution* $\{\underline{u}_{i,n}, \underline{v}_{i,n}\}$ *of discrete time degenerate Dirichlet IBVP* (5)-(7). *More over maximal solution* $\{\bar{u}_{i,n}, \bar{v}_{i,n}\}$ *and minimal solution* $\{\underline{u}_{i,n}, \underline{v}_{i,n}\}$ *satisfy the relation,*

$$(\hat{u}_{i,n}, \hat{v}_{i,n}) \leq (\underline{u}_{i,n}, \underline{v}_{i,n}) \leq (\bar{u}_{i,n}, \bar{v}_{i,n}) \leq (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \text{ in } \bar{\Lambda}_p \tag{14}$$

Proof. From Lemma 4.1, we conclude that the sequence $\{\bar{u}^{(m)}_{i,n}, \bar{v}^{(m)}_{i,n}\}$ is monotone nonincreasing and bounded from below hence it converges to limit function say $(\bar{u}_{i,n}, \bar{v}_{i,n})$. Also the sequence $\{\underline{u}^{(m)}_{i,n}, \underline{v}^{(m)}_{i,n}\}$ is monotone nondecreasing and is bounded from above hence it converges to limit function say $(\underline{u}_{i,n}, \underline{v}_{i,n})$. So the following limits:

$$\lim_{m\to\infty} \bar{u}^{(m)}_{i,n} = \bar{u}_{i,n},$$
$$\lim_{m\to\infty} \bar{v}^{(m)}_{i,n} = \bar{v}_{i,n},$$
$$\lim_{m\to\infty} \underline{u}^{(m)}_{i,n} = \underline{u}_{i,n},$$
$$\lim_{m\to\infty} \underline{v}^{(m)}_{i,n} = \underline{v}_{i,n}$$

exist and called maximal and minimal solutions respectively of the discrete time degenerate Dirichlet IBVP (5)-(7) and they satisfy the relation

$$(\hat{u}_{i,n}, \hat{v}_{i,n}) \leq (\underline{u}_{i,n}, \underline{v}_{i,n}) \leq (\bar{u}_{i,n}, \bar{v}_{i,n}) \leq (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \text{ in } \Lambda_p.$$

■

Theorem 5.2. (Uniqueness Theorem) *Suppose that*

(i) $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ *and* $(\hat{u}_{i,n}, \hat{v}_{i,n})$ *are ordered upper and lower solutions of discrete time degenerate Dirichlet IBVP* (5)-(7).

(ii) *the function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *is mixed quasimonotone in* $S_{i,n}$,

(iii) *the function* $(f^{(1)}_{i,n}, f^{(2)}_{i,n})$ *satisfies the following Lipschitz condition*

$$-c^{(1)}_{i,n}(u_{i,n} - u'_{i,n}) \leq f^{(1)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(1)}_{i,n}(u'_{i,n}, v_{i,n}) \quad \leq c^{(1)}_{i,n}(u_{i,n} - u'_{i,n})$$
$$\textit{when } u_{i,n} \geq u'_{i,n},$$
$$-c^{(2)}_{i,n}(v_{i,n} - v'_{i,n}) \leq f^{(2)}_{i,n}(u_{i,n}, v_{i,n}) - f^{(2)}_{i,n}(u_{i,n}, v'_{i,n}) \quad \leq c^{(2)}_{i,n}(v_{i,n} - v'_{i,n})$$
$$\textit{when } v_{i,n} \geq v'_{i,n}.$$

Then the discrete time degenerate Dirichlet IBVP (5)-(7) *has unique solution.*

Proof. We know that $(\bar{u}_{i,n}, \bar{v}_{i,n})$ and $(\underline{u}_{i,n}, \underline{v}_{i,n})$ are maximal and minimal solutions respectively of the discrete time degenerate Dirichlet IBVP (5)-(7). To prove the uniqueness of solution, it is sufficient to show that

$$\underline{u}_{i,n} \geq \bar{u}_{i,n} \textit{ and } \underline{v}_{i,n} \geq \bar{v}_{i,n}.$$

Define a function $w_{i,n} = \underline{u}_{i,n} - \bar{u}_{i,n}$. Observe that

$$\begin{aligned}
&d^{(1)}_{i,n} k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n} \triangle^{(\nu)} w_{i,n} \\
=&f^{(1)}_{i,n}(\underline{u}_{i,n}, \underline{v}_{i,n}) - f^{(1)}_{i,n}(\bar{u}_{i,n}, \bar{v}_{i,n}) \\
=&f^{(1)}_{i,n}(\underline{u}_{i,n}, \underline{v}_{i,n}) - f^{(1)}_{i,n}(\bar{u}_{i,n}, \underline{v}_{i,n}) + f^{(1)}_{i,n}(\bar{u}_{i,n}, \underline{v}_{i,n}) - f^{(1)}_{i,n}(\bar{u}_{i,n}, \bar{v}_{i,n}) \\
\geq& - \bar{c}_{i,n} w_{i,n} + f^{(1)}_{i,n}(\bar{u}_{i,n}, \underline{v}_{i,n}) - f^{(1)}_{i,n}(\bar{u}_{i,n}, \bar{v}_{i,n}).
\end{aligned}$$

Since $f^{(1)}_{i,n}$ is mixed quasimonotone, we obtain

$$\begin{aligned}
&d^{(1)}_{i,n} k_n^{-1}(w_{i,n} - w_{i,n-1}) - \sum_{\nu=1}^{p} D^{(1)}_{i,n} \triangle^{(\nu)} w_{i,n} + \bar{c}_{i,n} w_{i,n} \geq 0 \\
&w_{i,n} \geq 0 \\
&w_{i,0} \geq 0.
\end{aligned}$$

Applying Lemma 3.3, we get $w_{i,n} \geq 0$ i.e. $\underline{u}_{i,n} \geq \bar{u}_{i,n}$ and we conclude that $\underline{u}_{i,n} \equiv \bar{u}_{i,n}$. On similar lines, we can prove that $\underline{v}_{i,n} \geq \bar{v}_{i,n}$ and we conclude that $\underline{v}_{i,n} \equiv \bar{v}_{i,n}$. Hence the result. ■

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